Accelerated CCGPS Pre-Calculus
Unit 3: Trigonometry of General Triangles
Unit 3

Trigonometry of General Triangles

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OVERVIEW
In this unit students will:
- Expand the use of trigonometric functions beyond right triangles into more general triangles.
- Develop the trigonometric formula for area of triangle.
- Use the Laws of Sines and Cosines to solve problems.

Although the units in this instructional framework emphasize key standards and big ideas at specific times of the year, routine topics such as estimation, mental computation, and basic computation facts should be addressed on an ongoing basis. Ideas related to the eight practice standards should be addressed constantly as well. To assure that this unit is taught with the appropriate emphasis, depth, and rigor, it is important that the tasks listed under “Evidence of Learning” be reviewed early in the planning process. A variety of resources should be utilized to supplement this unit. This unit provides much needed content information, but excellent learning activities as well. The tasks in this unit illustrate the types of learning activities that should be utilized from a variety of sources.

STANDARDS ADDRESSED IN THIS UNIT

Mathematical standards are interwoven and should be addressed throughout the year in as many different units and activities as possible in order to emphasize the natural connections that exist among mathematical topics. Students will build on their previous experience with trigonometric functions to include applications of the Laws of Sines and Cosines.

KEY STANDARDS

Apply trigonometry to general triangles

MCC9-12.G.SRT.9 (+) Derive the formula $A = (1/2)ab \sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.

MCC9-12.G.SRT.10 (+) Prove the Laws of Sines and Cosines and use them to solve problems.

MCC9-12.G.SRT.11 (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

RELATED STANDARDS

Define trigonometric ratios and solve problems involving right triangles

MCC9-12.G.SRT.6 Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.
MCC9-12.G.SRT.7 Explain and use the relationship between the sine and cosine of complementary angles.

MCC9-12.G.SRT.8 Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.

STANDARDS FOR MATHEMATICAL PRACTICE

The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important “processes and proficiencies” with longstanding importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council’s report Adding It Up: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one’s own efficacy).

1. Make sense of problems and persevere in solving them. High school students start to examine problems by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. By high school, students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. They check their answers to problems using different methods and continually ask themselves, “Does this make sense?” They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

2. Reason abstractly and quantitatively. High school students seek to make sense of quantities and their relationships in problem situations. They abstract a given situation and represent it symbolically, manipulate the representing symbols, and pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Students use quantitative reasoning to create coherent representations of the problem at hand; consider the units involved; attend to the meaning of quantities, not just how to compute them; and know and flexibly use different properties of operations and objects.
3. **Construct viable arguments and critique the reasoning of others.** High school students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. High school students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. High school students learn to determine domains to which an argument applies, listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

4. **Model with mathematics.** High school students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. High school students making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5. **Use appropriate tools strategically.** High school students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. High school students should be sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. They are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

6. **Attend to precision.** High school students try to communicate precisely to others by using clear definitions in discussion with others and in their own reasoning. They state
the meaning of the symbols they choose, specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

7. **Look for and make use of structure.** By high school, students look closely to discern a pattern or structure. In the expression \( x^2 + 9x + 14 \), older students can see the 14 as \( 2 \times 7 \) and the 9 as \( 2 + 7 \). They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see \( 5 - 3(x - y)^2 \) as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers \( x \) and \( y \). High school students use these patterns to create equivalent expressions, factor and solve equations, and compose functions, and transform figures.

8. **Look for and express regularity in repeated reasoning.** High school students notice if calculations are repeated, and look both for general methods and for shortcuts. Noticing the regularity in the way terms cancel when expanding \( (x - 1)(x + 1) \), \( (x - 1)(x^2 + x + 1) \), and \( (x - 1)(x^3 + x^2 + x + 1) \) might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, derive formulas or make generalizations, high school students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

**Connecting the Standards for Mathematical Practice to the Standards for Mathematical Content**

The Standards for Mathematical Practice describe ways in which developing student practitioners of the discipline of mathematics should engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle and high school years. Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction.

The Standards for Mathematical Content are a balanced combination of procedure and understanding. Expectations that begin with the word “understand” are often especially good opportunities to connect the practices to the content. Students who do not have an understanding of a topic may rely on procedures too heavily. Without a flexible base from which to work, they may be less likely to consider analogous problems, represent problems coherently, justify conclusions, apply the mathematics to practical situations, use technology mindfully to work with the mathematics, explain the mathematics accurately to other students, step back for an overview, or deviate from a known procedure to find a shortcut. In short, a missing mathematical knowledge effectively prevents a student from engaging in the mathematical practices.
In this respect, those content standards which set an expectation of understanding are potential “points of intersection” between the Standards for Mathematical Content and the Standards for Mathematical Practice. These points of intersection are intended to be weighted toward central and generative concepts in the school mathematics curriculum that most merit the time, resources, innovative energies, and focus necessary to qualitatively improve the curriculum, instruction, assessment, professional development, and student achievement in mathematics.

**ENDURING UNDERSTANDINGS**

- Derive a formula for the area of a triangle using two sides and a non-included angle
- Calculate the area of a general triangle using \( A = \frac{1}{2} ab \sin C \)
- Apply the Law of Sines and the Law of Cosines to solve problems.

**CONCEPTS/SKILLS TO MAINTAIN**

It is expected that students will have prior knowledge/experience related to the concepts and skills identified below. It may be necessary to pre-assess in order to determine if time needs to be spent on conceptual activities that help students develop a deeper understanding of these ideas.

- number sense
- computation with whole numbers and decimals, including application of order of operations
- calculating the area of a triangle
- solving trigonometric equations
- using inverse trigonometric functions to solve problems
- constructing altitudes in a triangle
- performing operations with trigonometric functions

**SELECTED TERMS AND SYMBOLS**

The following terms and symbols are often misunderstood. These concepts are not an inclusive list and should not be taught in isolation. However, due to evidence of frequent difficulty and misunderstanding associated with these concepts, instructors should pay particular attention to them and how their students are able to explain and apply them.

*The definitions below are for teacher reference only and are not to be memorized by the students.* Students should explore these concepts using models and real life examples. Students should understand the concepts involved and be able to recognize and/or demonstrate them with words, models, pictures, or numbers.

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**Georgia Department of Education**  
Common Core Georgia Performance Standards Framework Student Edition  
*Accelerated CCGPS Pre-Calculus • Unit 3*

Dr. John D. Barge, State School Superintendent  
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The website below is interactive and includes a math glossary.

http://intermath.coe.uga.edu/dictionary/homepg.asp

Definitions and activities for these and other terms can be found on the Intermath website. Links to external sites are particularly useful.

- **Altitude of a Triangle**: The perpendicular distance between a vertex of a triangle and the side opposite that vertex. Sometimes called the height of a triangle. Also, sometimes the line segment itself is referred to as the altitude.

- **Hinge Theorem**: If two sides of one triangle are congruent to two sides of another triangle, and the included angle of the first is larger than the included angle of the second, then the third side of the first triangle is longer than the third side of the second triangle. (Wikipedia)

- **Included Angle**: The angle between two given sides of a triangle

- **Law of Cosines**: The square of the length of any side of a triangle equals the sum of the squares of the lengths of the other two sides minus twice the product of the lengths of the other two sides and the cosine of the angle between them. (Swokowski, Cole)

- **Law of Sines**: In any triangle, the ratio of the sine of an angle to the side opposite that angle is equal to the ratio of the sine of another angle to the side opposite that angle (Swokowski, Cole)

- **Oblique Triangle**: A triangle that is not a right triangle

- **Vertex of a Triangle**: The common endpoint of the two legs that serve as the sides of a triangle
FINDING A NEW AREA FORMULA FOR TRIANGLES LEARNING TASK

Common Core Standards:

MCC9-12.G.SRT.9 -Derive the formula $A= \frac{1}{2}ab \sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side

Introduction:

This task is designed to help a student create a new formula for calculating the area of a triangle by combining the familiar formula for area with the trigonometric ratios that they have learned previously.
FINDING A NEW AREA FORMULA FOR TRIANGLES LEARNING TASK

A developer needs to find the area of some plots of land he is interested in buying. Each plot is owned by a different person and neither owner knows the actual area of the land. The diagram below illustrates the plots he wants to buy but he wants to know the area before buying it.

Your task is to calculate the total areas of the plots.

1. Recall that the formula for the area of a triangle is \( A = \frac{1}{2}bh \)
   where \( b \) is the length of the base and \( h \) is the height, perpendicular to the base. Can the formula be used in this situation? Why or why not?

2. What information would be helpful in determining the area of the triangles? (Hint: Can you draw another line that would help?)

3. Calculate the height of \( \triangle ABC \). Find the area. Use your strategy to calculate the area of \( \triangle ABD \). What is the total area of the two properties?
4. After working on this problem, would be possible to generalize this method for use on any triangle?

Construct any triangle. Label the angles A, B, and C. Label the side opposite from A with a, the side opposite B with b, and the side opposite C with c.

Construct any altitude from a vertex.

Now calculate the height of your triangle.

Now that you know the height of the triangle, you can write a general formula for the area of any triangle. Use $A = \frac{1}{2}bh$ as a starting point.

5. Compare your formula with another student. Did you get the same thing? Could you both be right?
6. Would it be possible to develop a third formula for the area? If so, find it. If not, explain why not.
PROVING THE LAW OF COSINES

Common Core Standards

MCC9-12.G.SRT.10 Prove the Laws of Sines and Cosines and use them to solve problems.

MCC9-12.G.SRT.11 Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles

Introduction

The purpose of this task is to guide students through the derivation of the Law of Cosines. The teacher should spend the extra time and effort in helping students understand the conceptual foundation for the Law of Cosines, and not just memorizing the formula. A few problems have been provided at the end of the task, but the classroom teacher should provide ample opportunities to practice and improve fluency.
PROVING THE LAW OF COSINES

During a baseball game an outfielder caught a ball hit to
dead center field, 400 feet from home plate. If the distance
from home plate to first base is 90 feet, how far does the
outfielder have to throw the ball to get it to first base?

1. Model the problem with a picture. Be sure to label
   information that you know.

2. Do you have enough information to solve the problem? If not, what is missing?

Typically, you have solved triangles that are right triangles. This is a case where we do not have
a right triangle to solve. We know two sides and one included angle. In this task, you will
develop a method for solving triangles like this using trigonometry. We will come back to the
baseball example later. For now, consider the triangle below. Follow these steps to derive a way
to solve for \( c \) knowing just that much information. For this example, assume we know
measurements for segments \( a \), \( b \), and angle \( C \).

3. What does segment \( h \) represent? What are its properties and what does it do to the large
   triangle?
4. Write an equation that represents $c^2$. Explain the method you used.

5. Now write an equation that represents $h^2$ in terms of $b$ and $x$. Substitute this expression into the expression you wrote in #4. Expand and simplify.

6. Now write an expression that represents $x$ in terms of the angle $C$. Substitute this expression into the equation you wrote in #5. Simplify completely.

Your answer to #6 is one of three formulas that make up the **Law of Cosines**. Each of the formulas can be derived in the same way you derived this one by working with each vertex and the other heights of the triangle.

**Law of Cosines**

Let $a$, $b$, and $c$ be the lengths of the legs of a triangle opposite angles $A$, $B$, and $C$. Then,

\[
\begin{align*}
    a^2 &= b^2 + c^2 - 2bc \cos A \\
    b^2 &= a^2 + c^2 - 2ac \cos B \\
    c^2 &= a^2 + b^2 - 2ab \cos C
\end{align*}
\]

These formulas can be used to solve for unknown lengths and angles in a triangle.

7. Solve the baseball problem at the beginning of this task using the Law of Cosines.
Here are a few problems to help you apply the Law of Cosines.

8. Two airplanes leave an airport, and the angle between their flight paths is 40°. An hour later, one plane has traveled 300 miles while the other has traveled 200 miles. How far apart are the planes at this time?

9. A triangle has sides of 8 and 7 and the angle between these sides is 35°. Solve the triangle. (Find all missing angles and sides.)

10. Three soccer players are practicing on a field. The triangle they create has side lengths of 18, 14, and 15 feet. At what angles are they standing from each other?

11. Is it possible to know two sides of a triangle and the included angle and not be able to solve for the third side?
PROVING THE LAW OF SINES

Common Core Standards:

MCC9-12.G.SRT.10 Prove the Laws of Sines and Cosines and use them to solve problems.

MCC9-12.G.SRT.11 Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles

Introduction

The purpose of this task is to guide students through the process of deriving the Law of Sines. Emphasis should be placed on conceptual understanding rather than memorizing a formula.
PROVING THE LAW OF SINES

When using the Law of Cosines it is necessary to know the angle included between two sides. However, there are times the angle that is known is not the angle included between two known sides. And there are other times where we might know two angles and only one side. Consider the following triangle:

1. Construct an altitude through C. Label it \( h \).

2. Write an equation for \( h \) in terms of angle \( B \). Also write an equation for \( h \) in terms of angle \( A \).

3. Write a new equation relating the two equations together.

4. Now construct another altitude through \( B \). Label it \( j \).
5. Write an equation for \( j \) in terms of angle \( A \). Also write an equation for \( j \) in terms of angle \( C \).

6. Write a new equation relating the two equations together.

7. Using the Transitive Property of Equality, write an equation relating the equations in #3 and #6.

What you have arrived at is known as the Law of Sines.

### Law of Sines

Let \( a, b, \) and \( c \) be the lengths of the legs of a triangle opposite angles \( A, B, \) and \( C \). Then,

\[
\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}
\]

Here is an example of applying the Law of Sines. In the triangle to the right the measurements for angle \( B \) and \( A \) are missing and the length of \( AC \). Follow the steps below to solve for the measure using the Law of Sines.

This is the information that we know so far.

\[
\frac{6.7}{\sin 33} = \frac{5.4}{\sin A} = \frac{b}{\sin B}
\]

- Solve the first two ratios to find \( A \).

- Find angle \( B \). (Remember that the sum of the interior angles in a triangle is always 180 degrees)
• Use either the first or second ratio with the last ratio to solve for side b.

Here are some problems that will help you practice applying the Law of Sines. Begin each problem by drawing a picture to model the situation. Then solve the problem.

8. A surveyor is near a river and wants to calculate the distance across the river. He measures the angle between his observations of two points on the shore, one on his side and one on the other side, to be 28°. The distance between him and the point on his side of the river can be measured and is 300 feet. The angle formed by him, the point on his side of the river, and the point on the opposite side of the river is 128°. What is the distance across the river? Remember that the distance across the river should be the shortest distance.

9. Two people are observing a hot air balloon from different places on level ground. The angles of elevation are 20° and 50°. The two people are 7 miles apart and the balloon is between them. How high is the balloon off the ground?
THE HINGE THEOREM

Common Core Standards

MCC9-12.G.SRT.10 Prove the Laws of Sines and Cosines and use them to solve problems.

MCC9-12.G.SRT.11 Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles

Introduction:

Although the Hinge Theorem is not part of the standards, this task makes a connection to concepts already known by the student and the Law of Sines. This exploration works well with construction software if it is available.
THE HINGE THEOREM

From your previous math experience you know that the measures of two sides and a non-included angle will not necessarily work together to create a triangle. The Hinge Theorem is a geometric theorem that focuses on this idea. You may have explored this idea when you studied congruent triangles.

Consider the two triangles below. Given sides of 7 cm and 4.2 cm with a non-included angle of $30^\circ$, there are two triangles that can be created. This is why angle-side-side is not a congruency theorem for triangles.

In trigonometry we consider this to be the ambiguous case for solving triangles.

Looking at the two triangles above, why do you think this theorem might be called the Hinge Theorem?

What is the relationship between the measure of $\angle ABC$ in the first figure and $m\angle ABC$ in the second figure?

Consider triangle ABC to the right. Note that $m\angle A = 30^\circ$, $AC = 7$ cm and $BC = 3.5$ cm.

Are there still 2 possible triangles? Explain your answer.

Redraw triangle ABC so that $m\angle A = 30^\circ$, $AC = 7$ cm and $BC = 9$ cm. Are there still 2 triangles? Explain your answer.

Redraw triangle ABC so that $m\angle A = 30^\circ$, $AC = 7$ cm and $BC = 2$ cm. Are there still 2 triangles? Explain your answer.
1. Order the sides of the triangle to the right from longest to shortest. (The figure is NOT drawn to scale.)

2. How did you decide the order of the sides?

3. Sketch the information given and decide if it is possible to create a triangle with the given information. Explain your answer.

Triangle 1: $A = 40^\circ$, $a = 8$ cm and $b = 5$ cm.

Triangle 2: $A = 150^\circ$, $a = 5$ cm and $b = 8$ cm

Be sure to take time to analyze the data you are given when you are solving a triangle to determine if you might have a set of data that will not yield a triangle. This could save you a lot of work!
Remember that the calculator only yields inverse sine values between $-90^\circ$ and $90^\circ$, so it will never let you know if there are obtuse angles in your triangles.

4. Solve the following triangle. Determine if there is no solution, one solution or two solutions.

Measure of angle $A = 35^\circ$, $a = 10$ cm and $b = 16$ cm

5. A ship traveled 60 miles due east and then adjusted its course 15 degrees northward. After traveling for a while the ship turned back towards port. The ship arrived in port 139 miles later. How far did the ship travel on the second leg of the journey? What angle did the ship turn through when it headed back to port?